Mental Computation, Number Sense and General Mathematics Ability: Are they Linked?

Alistair McIntosh
University Of Tasmania
<alistair.mcintosh@utas.edu.au>

Shelley Dole
University of Tasmania
<shelley.dole@utas.edu.au>

Number sense is generally regarded as an important aspect of general mathematics knowledge, while mental computation performance is frequently thought to link to number sense ability. With current national numeracy testing placing an emphasis on number sense, this study explores the relationship between number sense, mental computation and general mathematics performance of primary students. Results indicate that performance in these three areas was often disparate, and raise some interesting questions about number sense and mental computation and assessment of mathematics curriculum.

Introduction

Number Sense and Mental Computation

In the video *Real Maths, School Maths* (Lovitt & Clarke, 1992) a child (Aidan) aged about 10, is seen dealing with two computations. First, he is shown a chocolate bar and is told it costs forty-five cents; he is asked to calculate mentally what change he would get, if he bought one chocolate bar with a five-dollar note. He handles the calculation confidently and efficiently, and explains his strategy clearly. On a second occasion he is seen seated at a desk with paper on which the identical subtraction (500 - 45) is written in vertical form. Aidan expresses doubt whether it is a subtraction or a division, looks uncomfortable, performs the written subtraction incorrectly, and is at a loss to explain his method. Aidan's mental computation displays, and draws on, number sense that is markedly lacking in his written computation.

Although the term "number sense" is relatively new to the language of mathematics curricula, its emphasis on understanding and meaningful learning is commonplace in the literature of mathematics education (Brownell, 1935; Hiebert, 1984; Plunkett, 1979). Number sense refers to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for managing numerical situations (McIntosh, Reys & Reys, 1992). It is characterised by a desire to make sense of numerical situations, by looking for links between new information and previously acquired knowledge, and by an innate drive within the learner to make the forming of these connections a priority (Reys et al., 1991). Those with well-developed number sense have strongly connected knowledge and in learning they appreciate the importance of making links between new and previously acquired knowledge. They approach problems involving number by seeking to make sense of the whole situation and by using the multiple representations and connected knowledge that they have.

Mental computation, or the ability to conduct mental numerical calculations without recourse to external assistance (e.g., pen and paper), is also regarded as a valuable skill for all Australians (Australian Education Council, 1991). Research has indicated that people use a variety of (usually self-devised) strategies to assist mental computation (e.g., Bana & Korbosky, 1995; Swan, 1991; McIntosh, De Nardi, & Swan, 1994). There is evidence to suggest that at present students are often not directly exposed to such strategies at school but rather are left to devise for themselves more or less efficient strategies (McIntosh, Bana, & Farrell, 1995). Such strategies appear to derive from an individual's flexibility in handling numbers, and the development of useful strategies applied in efficient ways. Strategy development for mental computation, therefore, can be seen to derive to some degree from an individual's number sense.

National Numeracy Benchmarks

National standardised testing, particularly in Grade 3 and Grade 5, is a feature of the current school situation in Australia (Cuttance & Stokes, 2000), born out of the national education goal that children will be literate and numerate upon exiting primary school (MCEETYA, 1999). National literacy and numeracy benchmarks are being developed to provide a framework for reporting student performance at particular levels (Curriculum Corporation, 1999). The National Numeracy Benchmarks are categorised into 3 strands of Number Sense, Measurement and Data Sense, and Spatial Sense, and are incorporated into the standardised tests for numeracy developed in each state (Cuttance & Stokes, 2000). There is clearly a growing emphasis on the development of number sense within the mathematics curriculum. Indeed, documents calling for reform of school mathematics in many industrialised countries have emphasised the need for students to develop number sense (e.g., Cockcroft, 1982; Emanuelsson & Johansson, 1996; National Council of Teachers of Mathematics, 1989).

Implications for this Study

The advent of national testing of literacy and numeracy in the primary school (currently Grade 3 and Grade 5) and the development of National Numeracy Benchmarks that place an emphasis on "sense" about number, space, measurement and data, give rise to a number of questions pertaining to comprehensiveness, reliability and accuracy in determining children's mathematics ability. Primarily, the question is, would a general mathematics test be sufficient to assess a student's number sense as defined in the literature? A further dimension to mathematics performance and number sense occurs if mental computation is also considered. If mental computation performance is enhanced by number sense, then where, if at all, does the assessment of mental computation occur within assessment practices at school, system or national level? The exploration of such issues was the basis for this study.

This study was designed to begin to probe the relationship between the mental computation ability, number sense, and general mathematics of Grade 3 and Grade 5 students, and to raise questions about assessing mental computation and number sense, particularly in the current climate of increased emphasis on system-wide assessments and of national testing in Grade 3, Grade 5 and beyond. The following exploratory questions were posed at the outset of this research: Is mental computation ability closely linked with number sense, as some believe, but more loosely connected with general mathematical ability? Is a mental computation test assessing something different from a general mathematics test or a test of number sense, or will a class of children or individual children perform equally well on each?

The Study

Separate pencil-and-paper tests of mental computation (MC), number sense (NS) and general mathematics (MA) were given to a total of 58 Year 3 and 60 Year 5 students in two Tasmanian primary schools. Tests were administered during students' usual mathematics time over three days. The mental computation and number sense tests each contained thirty items at Grade 3 level and forty items at Grade 5 level. The general mathematics tests were devised by taking items from Diagnostic Mathematics Profiles (Gough 1999). Validity of the two general mathematics tests was obtained by subjecting them to the scrutiny of two experts and by the agreement of the four class teachers involved that the tests reflected the range and level of mathematics being taught in their classrooms. Reliability tests (Cronbach-Alpha) for internal consistency for the Grade 3 and Grade 5 tests of mental computation, number sense, and general mathematics gave the following results: Grade 3 - MC 0.91, NS 0.68, MA 0.75; Grade 5 - MC 0.93, NS 0.84, MA 0.90. Apart from the Grade 3 number sense test, and to some extent the Grade 3 Mathematics tests, the tests appear to be extremely reliable. The difficulty of constructing adequate pencil-and-paper tests of number sense at grades as low as Grade 3 has been discussed elsewhere (McIntosh, Reys, Reys, Bana & Farrell, 1997).

On the basis of test results, students were selected for individual interviews to explore their solution strategies and thinking. Students were chosen either because their results on the three tests were markedly similar, or because they were markedly dissimilar.

Results and Discussion

For Grade 3, the mean scores for MC, NS and MA were 78%, 67% and 77% respectively. For Grade 5, the mean scores for MC, NS and MA were 65%, 58% and 67% respectively. In Table 1, mean scores for each of the tests, together with standard deviations, minimum and maximum scores are presented.

Table 1

Summary of Results as Percentages for Grade 3 (n=58) and Grade 5 (n=60) on the Mental Computation (MC), Number Sense (NS) and Mathematics (MA) Tests

	Grade 3				Grade 5			
	x	SD	min	max	x	SD	min	max
MC	78	20	20	100	65	24	3	97
NS	67	16	20	95	58	19	0	93
MA	77	15	21	96	67	18	20	93

Correlation tests (by Fisher's r to z) showed that the scores on each of the three tests were closely associated with each other. To ascertain the extent to which performance on one test predicted performance on another, simple regressions (r^2) were calculated. The results are shown in Table 2.

At both levels there appears to be an appreciable amount of the variance in the mental computation scores (Grade 3: 0.36, Grade 5: 0.33) which is not predicted by the mathematics scores. This is even more marked in the case of the number sense scores (Grade 3: 0.56, Grade 5: 0.45). While it is not possible to make confident assertions on the basis of these very small samples, MA scores appear to be a better predictor of MC than NS for both grade levels, and MC scores appear to be a better predictor of NS at Grade 5 than at Grade 3. This result is not surprising given the fact that the Grade 3 NS test had a much lower reliability score than that for Grade 5 NS.

Table 2
Simple Regressions (r^2) at Grade 3 (n=58) and Grade 5 (n=60) between Scores on the Mental Computation (MC), Number Sense (NS) and Mathematics (MA) tests

	MA:MC	MA:NS	MC:NS
Grade 3	0.64	0.44	0.59
Grade 5	0.67	0.55	0.72

Interviews

Eight students from each Grade level were selected for individual interview on the basis of their test scores. For brevity, interview data for three Grade 5 students (Student 5A, 5B and 5C) and three Grade 3 students (Student 3A, 3B and 3C) will be presented here.

Grade 5 Students

Student 5A scored highly on both the mental computation (91%) and general mathematics (93%) tests, but relatively low on the number sense test (67%). Interview data revealed that, for mental computation, this student executed written algorithms mentally at a very rapid rate, and the written algorithm performed mentally was the predominant strategy used. For example, when asked to explain her strategy for mentally calculating 72 ÷ 3, the student replied: "I put down a division sort of thing, well how you write down on paper, how you work it out with that kind of one and the curvy bit and I just did it in my head instead of doing it on paper." When asked to elaborate on the process, the student replied: "I had 72 under the line and then there was 3...I went 3 into 7 is twice and I put down the 1 next to the 2...and ...3 into 12 is 4 so 24."

This response indicates the student's capacity to hold a great deal of symbolic information in her head at once, and to visualise the written operation to achieve the correct answer. Interestingly, the student had difficulty dealing with certain number situations if she had not previously encountered them, evident in her response when she was asked by the researcher to mentally calculate 450 ÷ 15: "Okay, I haven't learnt these yet...I don't know, I haven't done double numbers divided."

However, when given the "double numbers divided" problem of 3500 ÷ 35, the student immediately noticed the relationship between the numbers by stating that the answer was 100 because: "... 3500 divided by 35, well they're sort of the same number so I took off ... 2 zeros is 100 so ... 100 times 35 is 3500". For items relating to number sense, there appears to be evidence that Student 5A has not become aware of the effects of some operations. For example, one of the number sense items asked students to determine whether, upon calculating 29 ÷ 0.8, the solution would be less than 29, equal to 29, greater than 29, or impossible to tell without calculating. Student 5A's response was as follows: "Well...in the test I kind of wasn't really sure and put the last answer" [D. Impossible to tell without calculating].

Another item on the number sense test related to estimation with students asked to choose the best estimate for 45 x 105 from three choices (4000, 4600, 5200). The following transcript appears to suggest that Student 5A is reluctant to use (or at least to articulate the use of) estimation as a strategy to assist with calculation. (In the transcript, the letters S and R are used to identify the dialogue of the student and researcher respectively).

S: Probably about 4600 or 5200 or something
R: So either of those two?
S: Yeah, maybe
R: Can you be more specific or would you rather work it out?

S: I'd rather work it out.

On a similar item, where students were asked to estimate 18 x 19, Student A2 replied:

S: Probably about one hundred and eighty

R: ...and what are you doing in your head?

S: Just guessing.

From interview questions relating to items on the general mathematics test, it appeared that Student 5A's understanding of fractions, measurement, space and chance and data assisted her to do well on this test. She also executed written algorithms flawlessly. However, on one particular item where students were asked to share $42 equally between 4 people, Student 5A totally disregarded the context of the question and simply performed the standard written division algorithm, as indicated by her response to her solution:

S: For this one I did 42 divided by 4 and I know my times tables so it'd be 10 and 2

R 10 remainder 2?

S: Yes

In Student 5A's case, there appears to be a high link between mental computation ability and mathematics ability. However, Student 5A's competent calculation skills, both written and mental, appear to have overridden any need for this student to explore the effects of operations on numbers. The student does have number sense in terms of "seeing" particular relationships between numbers, but other aspects of number sense are not evident.

Student 5B scored highly on all three tests, but her number sense test score was the highest (90%) for all Grade 5s tested (MC 89%, MA 97%). Interview data revealed that this student used a range of strategies for mental computation, assisted by being able to see relationships within and between numbers, as is evident from her solution processes to the following mental computation items. For 72 ÷ 3, Student 5B saw 72 as being a combination of 60 and 12, and explained her strategy as "I just did 3s into 60 and then how many 3s go into ...12 and then add them together." For 165 + 99, the student noticed the link of 99 to 100: "Well because 99 is close to 100 you just add that and when I've got the answer to that I'd take 1 off again". For items on the number sense test, the student could explain the effect of various operations on numbers, and could analyse solution strategies to particular calculations. This student's number sense clearly assisted her with mental calculations and number situations, but her less than perfect result on the mental computation test appears to be the result of inaccuracy in application of some strategies.

Student 5C scored at about the class average for mental computation and number sense (74% and 70% respectively), but had a relatively low general mathematics test score (62%). Interview data revealed that this student used visualised written algorithms for mental computation involving decimals, multiplication and division, but with less accuracy than Student 5A. This student also appeared to rely on "rules" for particular mental calculations (e.g., 3500 ÷ 35 = 100 because "35 only goes into 35 once with the two zeros on the end"; and 4200 ÷ 60 = 700 because " 6 goes into 42...7 times and then add the zeros which is 700"). In the number sense test, this student could explain the effect of operations on numbers and could analyse strategies for calculations. On the general mathematics test, the student's poor fraction knowledge appeared to hinder high performance, and also there was evidence of limited understanding of measurement concepts associated with mass. In this instance, the student's mental computation ability does not appear to be related to her number sense ability as the student does not apply a range of strategies for mental computation. The similarity in scores on each of these two tests actually appears to be quite unrelated when the interview data are analysed.

Grade 3 Students

Like Student 5A, Student 3A's mental computation and general mathematics scores were similar (90% and 83% respectively), but number sense score was relatively low (60%). Student 3A was found to have some very good mental strategies, for example in describing his solution process for 74 - 30, he stated "I just took 4 away ... 40 and then I put the 4 back on so that was 44". In the number sense test, there was some evidence of difficulty with strategy analysis for estimation (he was unable to identify a strategy from a given list that would determine whether correct change had been given). Interview questions pertaining to items from the general mathematics test revealed some good conceptual understanding of fractions, measurement and space.

Like Student 5B, Student 3B's MC and NS scores were similar, but MA score was considerably different. Unlike Student 5B, Student 3B's test scores indicated a weaker mental computation and number sense (both 50%), but a relatively high mathematics ability (88%). Interview data revealed that, when performing mental computation, this student saw possible solution strategies, but tended to employ a sometimes inefficient and inaccurate counting on and counting back strategy. For example, in the following interview extract, the student is describing her solution process for 36 - 9:

S: I counted...take 9 from 36

R: You were thinking very hard then, can you just tell me what was happening in your mind when you decided it was 27?

S: I was counting, but I was thinking of another way of doing it

R: Were you counting backwards?

S: Yes

R: How did you know when to stop?

S: I was using my fingers

R: And the other strategy you were thinking of?

S: You can take 10 equals 26 and...add on 1 equals 27

In the number sense test, Student 3B could describe appropriate strategies for checking calculations, but had difficulty on items relating to estimating sizes of collections and numeration. Items relating to the general mathematics test indicated good conceptual understanding of concepts associated with measurement, space and chance and data.

Like Student 5C, Student 3C scored the highest on NS for all Grade 3 students (95%), and her mental computation and general mathematics test also were high (87% and 92%) respectively. Interview data revealed that this student employed a wide range of strategies for mental computation, and knew about the effects of operations on numbers. Her strong conceptual knowledge assisted with general mathematics test items. The profile of this student is similar to Student 5C, where the strong sense of number relationships (an aspect of number sense) appears to lead to the application of a variety of mental strategies for mental computation. Also like Student 5C, Student 3C did not rate the highest on the mental computation test, but the level of "sense about numbers" related to high scores on both the number sense test and the general mathematics test.

The Grade 5 and Grade 3 children who performed well on tests of both mental computation and general math but not so well on tests of number sense appear to have developed useful and efficient strategies for managing mental situations, which is a component of number sense (McIntosh, Reys, & Reys, 1992), but appear not to have a general understanding of number and operations (McIntosh, Reys & Reys, 1992) or an awareness of the relative effect of operating on numbers (NCTM, 1989). These students' conceptual knowledge of various topics within the mathematics curriculum assisted in high scores for general mathematics tests. It is tantalising to suggest that high levels of competence in accuracy for mental computation exercises and general mathematics tasks has negated the need for these students to develop thinking about the processes of computations and operations.

The Grade 5 and Grade 3 students that scored the highest on the number sense test were seen to utilise a range of strategies for mental computation, (although their mental computation test scores were not the highest of all subjects in this study), and they appeared to have a greater awareness of numbers and the effects of operations on numbers. There appears to be a link between number sense and mental computation in terms of strategy knowledge. For the Grade 5 and Grade 3 students with similar MC and NS scores but relatively different MA scores, there appear to be other factors at work impeding their accuracy in mental computation (such as poor application of strategies, difficulty with holding a lot of numbers in the mind to complete calculations mentally, lack of confidence with large or unfamiliar numbers), and also performance on general mathematics tests (such as poor conceptual knowledge of the topic, or misconceptions).

Concluding Comments

This was an exploratory study to consider some aspects of the relationship between school mathematics, mental computation and number sense. It is not possible to generalise from the results of such a small sample. Nevertheless some interesting points for discussion about mental computation, number sense and general mathematics ability can be raised.

In this study, it appeared that the three tests were testing somewhat different things. Students' results on the three tests often were disparate, and in some cases quite remarkably so. Tentative results reported here tend to suggest that the three tests are not as closely connected as might have been expected. If all three areas (general mathematics, mental computation and number sense) are considered important, indeed essential, parts of mathematics at school, then it appears that mental computation and number sense need to become integral components of curriculum and assessment procedures, at class, school and system levels. Otherwise, the curriculum may be distorted by playing down the importance of number sense and mental computation, and students may be either advantaged or disadvantaged if there is failure to assess important aspects of mathematics.

From interview data, it was apparent that students with good mental strategies were not always the students that performed the highest on mental computation, but that often they had a good understanding or "sense" of numbers and their relationships. In all cases, good conceptual understanding of particular mathematics topics was seen to contribute to high scores on the general mathematics test, and good conceptual understanding assisted in understanding particular number sense and mental computation items (particularly fraction items).

In conclusion, this study suggests that students who may score highly on mental computation tests and general mathematics tests may not be developing a "sense" of numbers. And students who do not score highly on written tests of mental computation, number sense and general mathematics may still have quite good strategies for mental computation and a lot of "sense" about numbers. Fostering the development of number sense and conceptual understanding of numbers and operations, and probing further to ascertain whether a student's accuracy in mental computation is the result of successful mental application of written strategies or based on a more flexible range of mental strategies, may strengthen the relationship between mental computation, number sense and general mathematics ability.

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